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LETTER TO THE EDITOR

Non-unitarity in rational conformal field theories

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Abstract. We study the classification problem of rational conformal field theories without imposing unitarity. The only constraint which remains is that of modular covariance. This can be imposed by several related but not always equivalent conditions. We choose what we think is the most natural one, namely that the characters transform according to a finite-dimensional representation of the modular group. Our main result is that, in the case where the chiral algebra is reduced to the Virasoro algebra alone, the set of modular covariant theories contains only the minimal models of Belavin, Polyakov and Zamolodchikov.

It is certainly true that unitarity is a cornerstone of quantum theories, and in the case of two-dimensional conformal field theories (CFT) it has been exploited with great success [1] as one of the main constraints on the set of consistent models. However in certain circumstances it is necessary to relax this criterion and allow for non-unitarity, e.g. to be able to consider some models of statistical mechanics such as the Lee–Yang singularity [2]. The point of view of the present paper is that unitarity is not a very natural condition when undertaking the task of classifying rational CFT. Indeed the recent results in this direction [3] never depend on unitarity.

The only constraint which remains is then that the theory be well defined on a two-dimensional Riemann surface, which should reflect itself into nice modular transformation properties. Here we will restrict ourselves to tori for simplicity. Let us be more precise and start by defining the CFT we have in mind. First we fix an infinite-dimensional algebra A (having the Virasoro algebra vir as a subalgebra) of operators in a vector space E, consisting of a *finite* number of *irreducible* representations W_i, $i=0, 1, \ldots, N$, of A (A-modules). This defines what we will call a (chiral) rational CFT. Let c be the central charge of vir acting on E, τ the modular parameter with $\mathrm{Im}\,\tau > 0$ and $q = \exp(2\pi i\tau)$, then the character of W_i is the holomorphic function

$$\chi_i(\tau) = q^{-c/24}\mathrm{Tr}_{W_i} q^{L_0}. \tag{1}$$

The modular group $G = SL_2(\mathbb{Z})$ acts on τ by the substitutions

$$\tau \mapsto g\tau = \frac{a\tau + b}{c\tau + d} \tag{2}$$

and thus it acts naturally on the characters

$$\chi_i(\tau) \mapsto (g\chi_i)(\tau) = \chi_i(g^{-1}\tau). \tag{3}$$

Further, let V denote the vector space spanned by the χ_i.

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The problem we want to study is to classify all rational CFT which satisfy the following condition:

(M) The action of G on V defined by (3) is a linear representation.

This condition is satisfied by all known unitary and non-unitary rational CFT.

We shall give a solution to this problem in two cases, when A consists of the Virasoro algebra only, and when A is the $N = 1$ superconformal algebra. The extension of our study to the more complicated case of affine Kac-Moody algebras† is currently under investigation [6]. Our interest in this problem arises from closely related considerations in a paper of Kac and Wakimoto [4]. In this letter we shall study the differences between (M) and the definitions of [4], then we shall give the solution to the classification problem in the two cases mentioned above and finally we shall draw some conclusions.

In their paper, Kac and Wakimoto [4], instead of (M), choose the condition

(M1) The χ_i are modular functions for the principal congruence subgroup $\Gamma(n)$ for some positive integer n.

This condition is also met by all known examples. We shall find it instructive to discuss yet another condition:

(M2) The χ_i are modular functions for some invariant subgroup H of G of finite index‡.

Here we recall that a modular function for a subgroup $\mathrm{H} \subset \mathrm{G}$ is an invariant under the substitutions $g\tau$ for all $g \in \mathrm{H}$. The principal congruence subgroup of level n is the invariant subgroup

$$\Gamma(n) = \{g \in \mathrm{G} \mid g = 1 \bmod n\} \tag{4}$$

where the congruence is elementwise. It is known [7] that there exists an infinite number of invariant subgroups H which do not contain any $\Gamma(n)$. These are called noncongruence groups.

Clearly (M1) implies (M2). However, in general one cannot establish that (M1) or (M2) implies (M) or the converse, or that (M2) implies (M1). Let us discuss now in more detail why we cannot prove these statements at least for the moment, without first solving the classification problems associated with the conditions (M), (M1) and (M2).

(M1) or (M2)$\Rightarrow$(M). H being an invariant subgroup of finite index in G, let M_{H} be the vector space of modular functions for H. M_{H} is finite dimensional. If $\chi_i \in M_{\mathrm{H}}$ and $g \in \mathrm{G}$ then $g\chi_i$ lies in M_{H} again, but we do not know if its lies in V.

(M2)$\Rightarrow$(M1). The q-expansions of the characters must have integral coefficients. When $\mathrm{H} = \Gamma(n)$ one knows that there exists a basis of M_{H} all of whose elements satisfy this constraint. However when H is a non-congruence group, examples are known for which such a basis does not exist [8]. The question is: is this true for all non-congruence groups?

† Note that in this case the characters will be functions not only of τ but also of z, which is in a Cartan subalgebra and transforms non-trivially under the modular group. For a non-unitary representation one cannot put $z = 0$, because in general the characters are undefined at this particular value. Nevertheless the problem of classifying modular covariant representations and their non-specialised characters has a well defined mathematical, if not physical, meaning. See [4, 5].

‡ Finite index means that G/H is finite.

(M)$\Rightarrow$(M2). This is equivalent to saying that the kernel of the representation is of finite index in G. This would follow from a conjecture of Grothendieck [9]† on finiteness of the monodromy groups in systems of linear differential equations.

We shall now prove the following theorem.

Theorem. Let A be the Virasoro algebra. Then the set of all rational CFT obeying (M) coincides with the minimal models of BPZ [10], i.e. W_i are irreducible representations with c and h given by

$$c = 1 - 6(p-q)^2/pq \tag{5}$$

$$h = [(rp - sq)^2 - (p-q)^2]/4pq \tag{6}$$

where p and q are positive, relatively prime integers and r, s are integers such that

$$1 \leqslant r \leqslant q-1 \qquad \text{and} \qquad 1 \leqslant s \leqslant p-1. \tag{7}$$

Before going on to the proof, let us comment on the significance of this result. A particular case of the theorem has been known for a quite a while: if one adds to (M) the assumption of unitarity, then one gets the unitary minimal models obtained by putting $q = p+1$ in (5) and (6). This unitary classification is due to [1, 11] and to Cardy [12] who showed that a unitary rational CFT has $c < 1$. The theorem implies that Cardy's result also holds for non-unitary rational CFT.

Proof of the theorem. Let us concentrate on one particular irreducible character $\chi = \chi_i$ for some i. If χ is to belong to a finite-dimensional representation of G, the orbit Gχ should span a finite-dimensional space. Denote by ν_r the character of a Verma module:

$$\nu_r = q^r/\eta(\tau) \tag{8}$$

where $r = h - (c-1)/24$. From the works of Feigin and Fuchs [13] we learn that the irreducible characters fall into three types.

(I) Those which are finite linear combinations of Verma module characters.

(IIa) Those which are infinite linear combinations of Verma modules with c, h given by (5) and (6) but r, s outside the rectangle (7).

(IIb) The characters of the minimal BPZ models mentioned in the statement of the theorem. These are also infinite linear combinations of Verma modules.

We now prove that the orbit Gχ spans an infinite-dimensional space except in case (IIb). This will follow from the fact that $G\nu_r$ is infinite dimensional. To show that, consider the subgroup B consisting of elements g_n where n is an integer and

$$g_n = \begin{pmatrix} 1 & 0 \\ n & 1 \end{pmatrix}. \tag{9}$$

One computes the action of B which is given, up to a constant phase, by

$$g_n \nu_r(\tau) = (1 - n\tau)^{-1/2} \eta(\tau)^{-1} \exp[2\pi i r\tau/(1 - n\tau)]. \tag{10}$$

The functions $g_n \nu_r$ for $n \in \mathbb{Z}$ and arbitrary r are linearly independent, which proves our claim. Therefore case (I) is already excluded.

† I thank A Kontsevitch for pointing out this reference to me.

For case (II*a*), Di Francesco *et al* [14] have explicitly written down the character formula. Here the relevant information is that

$$\chi = \text{finite sum of the } \nu + \text{a modular function} \tag{11}$$

where the modular function is for a congruence subgroup. The orbit of the modular function is, of course, finite, but the orbit of the first term is infinite as in (I), so this case is also excluded.

So we are left with the representations of type (II*b*). But it is known [15] that their characters transform according to (M). This concludes the proof of our theorem.

One can also see that the only irreducible characters of the Virasoro algebra which are modular functions are those of type (II*b*). Thus in this case the conditions (M), (M1) and (M2) are equivalent. In [4] the statement of the theorem with condition (M) replaced by (M1), and a supersymmetric analogue (see below) are given without proof.

When A is a superconformal $N = 1$ algebra, we have an analogue of the theorem, which states that the set of all rational superconformal field theories is the supersymmetrised version of the BPZ minimal models, i.e. the W_i are irreducible representations with

$$c = \frac{3}{2}\left(1 - \frac{2(p-q)^2}{pq}\right) \tag{12}$$

$$h = \frac{(rp - qs)^2 - (p-q)^2}{8pq} + \frac{1-2\varepsilon}{16}. \tag{13}$$

Here $\varepsilon = \frac{1}{2}$ (respectively 0) if A is the Neveu-Schwarz (respectively Ramond) algebra, $p = q \bmod 2$, $(p-q)/2$ and q are relatively prime, $r - s = (1-2\varepsilon) \bmod 2$ and r, s are as in (7). The proof of this is the same as in the ordinary (Lie) case, using character formulae from [16]. However, instead of B one has to use the subgroup of elements g_n with even n. Note that the unitary representations in (12) and (13) are obtained when $q = p + 2$.

We have a rather simple method for classifying all rational CFT with a fixed chiral algebra A. It assumes the knowledge of the modular transformation properties of all irreducible characters of A. (This means that it is probably impractical for all but the simplest extended algebras, for a representation theory has so far not been worked out in most cases, e.g. the W algebras [17].) Then the characters which are singled out are those which belong to finite-dimensional orbits of the modular group. The example of purely conformal (Virasoro) theories has been completely solved above, thereby giving a somewhat different perspective on the particular place occupied by the minimal BPZ theories among the other CFT.

But surely the deeper and more beautiful questions, which we have barely mentioned in this letter, i.e. concerning the coefficients of congruence and non-congruence modular functions and on the validity of Grothendieck's conjecture, should be studied both for their mathematical interest and their applications to CFT.

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